

The analytic index for proper, Lie groupoid actions

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ABSTRACT. Many index theorems (both classical and in noncommutative geometry) can be interpreted in terms of a Lie groupoid acting properly on a manifold and leaving an elliptic family of pseudodifferential operators invariant. Alain Connes in his book raised the question of an index theorem in this general context. In this paper, an analytic index for many such situations is constructed. The approach is inspired by the classical families theorem of Atiyah and Singer, and the proof generalizes, to the case of proper Lie groupoid actions, some of the results proved for proper locally compact group actions by N. C. Phillips.

1. Introduction

The objective of this paper is to prove the following theorem.

THEOREM 1.1. *Let G be a Lie groupoid for which the unit space G^0 is a proper, G -compact G -space. Let (X, p) be a G -manifold which is a fiber bundle over G^0 with compact smooth manifold Z as fiber and with structure group $\text{Diff}(Z)$. Let \tilde{E}, \tilde{F} be smooth G -vector bundles over X and $D = \{D^u\} : C^{0,\infty}(X, \tilde{E}) \rightarrow C^{0,\infty}(X, \tilde{F})$ be an invariant, continuous family of elliptic pseudodifferential operators on the fibers X^u of X . Then D determines a $(\mathbb{C}, C_{\text{red}}^*(G))$ -Kasparov module whose $K_0(C_{\text{red}}^*(G))$ -class is the analytic index of D .*

The motivation for this result comes from noncommutative geometry. In his book ([6, p.151]), Alain Connes comments that a number of index theorems both in classical and in noncommutative geometry are all special cases of the same index theorem for G -invariant elliptic operators D on a proper G -manifold X , G being a Lie groupoid. In this situation, the index of D belongs to $K_0(C^*(G))$. The present paper will establish this under certain (commonly satisfied) conditions. The approach to the analytic index of D in this paper is based on an adaptation of the equivariant Atiyah-Singer index theorem for families. There are, of course, other approaches to the analytic index in the context of groupoids, notably those using quasi-isomorphisms and the “deformation” approach using the tangent groupoid (e.g. [6, 13, 14, 16, 17, 21, 22]). However, these approaches apply (at the present time) only to the case where $X = G$, and in particular, do not cover the

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classical families index theorem. Further, the action of G on itself is rather special: for example, that action is free. In the present paper, no freeness assumptions of the action of G on X are made.

Another merit in the approach here is that it relates more readily to pseudodifferential operators and Fredholm theory and the well-established, detailed, classical theory. It also gives rise to a groupoid equivariant K-theory for TX and the prospect of defining a topological index in the same spirit as that of the classical families index theorem. This will be discussed elsewhere.

The author believes that the proof of this paper can be adapted to give the analytic index in the full generality of [6, p.151] and plans to discuss this elsewhere. The restrictions in the present paper are largely a matter of convenience, since they enable one to use existing theories to shorten and simplify the proof. In particular, the space X is assumed, as in the work of Atiyah and Singer ([2]), to be a manifold with compact fiber over the unit space G^0 of G . (The base space G^0 is not, however, assumed to be compact, but rather just G -compact and proper as a G -space.) This enables one to appeal, for example, to the results on pseudodifferential operators used in [2]. Another advantage of this framework is that it gives rise to a Hilbert G -bundle, and this allows the techniques used by N. C. Phillips in his development of equivariant K-theory to be adapted to produce the required Kasparov module. Further, we have used the reduced C^* -algebra $C_{red}^*(G)$ rather than $C^*(G)$, thus avoiding having to use the disintegration theorem for locally compact groupoids. So the index of D constructed here lies in $K_0(C_{red}^*(G))$.

The second section collects, for the convenience of the reader, some facts and definitions about locally compact and Lie groupoids, while the third discusses G -spaces and G -manifolds. The fourth section discusses families of elliptic pseudodifferential operators invariant under the action of a Lie groupoid G . Such a family D gives an invariant Fredholm morphism on a certain Hilbert bundle, and the final section shows that associated with this bundle is a Hilbert module, whose bounded and compact module maps correspond exactly to the invariant bounded and compact morphisms on the Hilbert bundle. This section is modelled on the approach of N. C. Phillips in his book [25] – Phillips effectively deals with the case where G is a transformation group groupoid. So the Fredholm morphism D translates over to a Fredholm map on a Hilbert module over $C_{red}^*(G)$, which, by a standard process, gives a Kasparov module whose KK -class is the analytic index of D .

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2. Preliminaries on groupoids

If X is a locally compact Hausdorff space, then $\mathcal{C}(X)$ is the family of compact subsets of X , $C(X)$ is the algebra of bounded continuous complex-valued functions on X , and $C_c(X)$ is the subalgebra of $C(X)$ whose elements have compact support. For a smooth manifold X , we define $C^\infty(X)$ to be the algebra of C^∞ – complex-valued functions on X , and $C_c^\infty(X)$ to be the algebra of functions in $C^\infty(X)$ with compact support.

A *groupoid* is most simply defined as a small category with inverses. Spelled out axiomatically, a groupoid is a set G together with a subset $G^2 \subset G \times G$, a product

map $m : G^2 \rightarrow G$, where we write $m(a, b) = ab$, and an inverse map $i : G \rightarrow G$, where we write $i(a) = a^{-1}$ and where $(a^{-1})^{-1} = a$, such that:

- (1) if $(a, b), (b, c) \in G^2$, then $(ab, c), (a, bc) \in G^2$ and

$$(ab)c = a(bc);$$

- (2) $(b, b^{-1}) \in G^2$ for all $b \in G$, and if (a, b) belongs to G^2 , then

$$a^{-1}(ab) = b \quad (ab)b^{-1} = a.$$

We define the *range* and *source* maps $r : G \rightarrow G^0$, $s : G \rightarrow G^0$ by setting $r(x) = xx^{-1}$, $s(x) = x^{-1}x$. The *unit space* G^0 is defined to be $r(G)(= s(G))$, or equivalently, the set of idempotents u in G . The maps r, s fiber the groupoid G over G^0 with fibers $\{G^u\}, \{G_u\}$, so that $G^u = r^{-1}(\{u\})$ and $G_u = s^{-1}(\{u\})$. Note that $(x, y) \in G^2$ if and only if $s(x) = r(y)$.

For detailed discussions of groupoids (including locally compact and Lie groupoids as below), the reader is referred to the books [13, 15, 22, 28]. Important examples of groupoids are given by transformation group groupoids and equivalence relations.

For the purposes of this paper, a *locally compact groupoid* is a groupoid G which is also a second countable, locally compact Hausdorff space for which multiplication and inversion are continuous. More generally, there is a notion of *locally compact groupoid* which does not require the Hausdorff condition. These arise quite often in practice, and a detailed discussion of them is given in [22]. The details of the paper can be made to work in the general case since local arguments suffice. However, for convenience, the locally compact groupoids considered in the paper are assumed to be Hausdorff.

Let G be a locally compact groupoid. Note that G^2, G^0 are closed subsets of $G \times G, G$ respectively. Further, since r, s are continuous, every G^u, G_u is a closed subset of G .

For analysis on a locally compact groupoid G , it is essential to have available a *left Haar system*. This is the groupoid version of a *left Haar measure*, though unlike left Haar measure on a locally compact group, such a system may not exist and if it does, it will not usually be unique. However, in many cases, such as in the Lie groupoid case below, there is a natural choice of left Haar system.

A left Haar system on a locally compact groupoid G is a family of measures $\{\lambda^u\}$ ($u \in G^0$), where each λ^u is a positive regular Borel measure on the locally compact Hausdorff space G^u , such that the following three axioms are satisfied:

- (1) the support of each λ^u is the whole of G^u ;
(2) for any $f \in C_c(G)$, the function f^0 , where

$$f^0(u) = \int_{G^u} f d\lambda^u,$$

belongs to $C_c(G^0)$;

- (3) for any $g \in G$ and $f \in C_c(G)$,

$$(2.1) \quad \int_{G^{s(g)}} f(gh) d\lambda^{s(g)}(h) = \int_{G^{r(g)}} f(h) d\lambda^{r(g)}(h).$$

The existence of a left Haar system on G has topological consequences for G – it entails that both $r, s : G \rightarrow G^0$ are open maps ([22, p.36]). For each $u \in G^0$, the measure λ_u on $G_u = (G^u)^{-1}$ is given by: $\lambda_u(A) = \lambda^u(A^{-1})$.

Jean Renault showed ([28]) that $C_c(G)$, the space of continuous, complex-valued functions on G with compact support, is a convolution $*$ -algebra with product given by:

$$(2.2) \quad f_1 * f_2(g) = \int_{G^{r(g)}} f_1(h) f_2(h^{-1}g) d\lambda^{r(g)}(h).$$

The involution on $C_c(G)$ is the map $f \rightarrow f^*$ where

$$(2.3) \quad f^*(g) = \overline{f(g^{-1})}.$$

As in the case of a locally compact group, there is a reduced C^* -algebra $C_{red}^*(G)$ (and a universal C^* -algebra $C^*(G)$) for the locally compact groupoid G . The reduced C^* -algebra can be defined as follows. For each $u \in G^0$, we first define a representation π_u of $C_c(G)$ on the Hilbert space $L^2(G, \lambda_u)$. To this end, regard $C_c(G)$ as a dense subspace of $L^2(G, \lambda_u)$ and define for $f \in C_c(G), \xi \in C_c(G)$,

$$(2.4) \quad \pi_u(f)(\xi)(g) = f * \xi \in C_c(G).$$

The reduced C^* -algebra-norm on $C_c(G)$ can then ([22, p.108]) be defined by:

$$\|f\|_{red} = \sup_{u \in G^0} \|\pi_u(f)\|.$$

The groupoid C^* -algebra $C^*(G)$ is the completion of $C_c(G)$ under its largest C^* -norm. In the paper, we will concentrate (for reasons of technical simplicity) on the reduced C^* -algebra of G .

A locally compact groupoid G is called a *Lie groupoid* ([15, 22, 26, 27]) if G is a manifold such that:

- (1) G^0 is a submanifold of G ;
- (2) the maps $r, s : G \rightarrow G^0$ are submersions;
- (3) the product and inversion maps for G are smooth.

Note that G^2 is naturally a submanifold of $G \times G$ and every G^u, G_u is a submanifold of G . (See [22, pp.55-56].) Note that the maps r, s are open maps. The dimension of all of the G^u, G_u is the same, and will be denoted by l .

There is a basis for the topology of a Lie groupoid that proves useful. Since r is a submersion, it is locally equivalent to a projection. So for each $x \in G$, there exists an open neighborhood V of x and a diffeomorphism $\psi : V \rightarrow r(V) \times W$ where W is an open subset of \mathbb{R}^l , which is fiber preserving in the sense that we can write $\psi(x) = (r(x), \phi(x))$ for some function ϕ . So if $u \in r(V)$, then the restriction of ψ to $G^u \cap V$ is a diffeomorphism onto W . We will call such an open set V a *basic* open set and write $V \sim r(V) \times W$.

A. Connes ([6, p.101]) discusses convolution on $C_c^\infty(G)$ in terms of density bundles, and this is canonical. However, since the bundles involved are trivial, we can replace the densities by a left Haar system $\{\lambda^u\}$ which is *smooth* in the sense that in terms of a basic open set V , the map $(u, w) \rightarrow d\lambda^u/d\lambda_W(u, w)$ is C^∞ and strictly positive on $r(V) \times W$. (Here, λ_W is the restriction of Lebesgue measure on \mathbb{R}^l to W .) For more details about this, see [22, 2.3]. All such smooth left Haar systems are equivalent in the obvious sense, and give the same $C_{red}^*(G)$ (and the same $C^*(G)$). For the rest of the paper, $\{\lambda^u\}$ will be a fixed smooth left Haar system on the Lie groupoid G .

3. G -spaces and G -manifolds

For the purposes of groupoid theory, we require the notions of proper G -spaces and G -manifolds (e.g. [20], [6, p.151]).

Let X, Y be locally compact, second countable topological spaces and $p : X \rightarrow Y$ be a continuous, open surjection. We will then say that (X, p) is *fibred over* Y . We will be particularly concerned with the case where $Y = G^0$, the unit space of a locally compact groupoid G . The space fibers of X are then the sets $X^u = p^{-1}(\{u\})$ ($u \in G^0$).

If (X_i, p_i) ($i = 1, 2$) are fibred over G^0 , then the fibred product $(X_1 * X_2, \tilde{p})$ is also fibred over G^0 , where

$$X_1 * X_2 = \{(x, y) \in X_1 \times X_2 : p_1(x) = p_2(y)\},$$

and $\tilde{p}(x, y) = p_1(x) = p_2(y)$.

Let $G * X$ be the fibred product of (G, s) with (X, p) . The pair (X, p) is defined to be a G -space if there is given a continuous map $m : G * X \rightarrow X$, where we write gx for $m(g, x)$, such that for $(g, x) \in G * X$ and $(h, g) \in G^2 (= G * G)$, we have:

- (1) $p(gx) = r(g)$;
- (2) $h(gx) = (hg)x$;
- (3) $g^{-1}(gx) = x$.

Of course, the pair (G, r) is a G -space under the groupoid multiplication. We note here that the unit space G^0 of G , as well as G itself, is a G -space. Here, the map p is just the identity map, and the action is given by:

$$g.s(g) = r(g).$$

If (X_i, p_i) ($i = 1, 2$) are G -spaces, then the fibred product $(X_1 * X_2, \tilde{p})$ is also a G -space under the action: $(g, x, y) \rightarrow (gx, gy)$ ($g \in G^{p(x)}$). For a G -space X , let $G *_r X = \{(g, x) : r(g) = p(x)\}$ be the fibred product of the G -spaces (G, r) and (X, p) .

Let (X, p) be a G -space. Then for each $g \in G$, the map $l_g : X^{s(g)} \rightarrow X^{r(g)}$, where $l_g(x) = gx$, is a homeomorphism, with inverse $l_{g^{-1}}$. Note also that if $u \in G^0$, then l_u is the identity map on X^u . Also, associated with the left action of G on X is a left action $g \rightarrow L_g$ of G on the bundle of algebras $C_c(X^u)$ ($u \in G^0$). For this, we define $L_g : C_c(X^{s(g)}) \rightarrow C_c(X^{r(g)})$ by:

$$L_g f(x) = f(g^{-1}x) \quad (x \in X^{s(g)}).$$

Clearly, each L_g is an isomorphism of $*$ -algebras, and both $(L_g)^{-1} = L_{g^{-1}}$ and $L_{gh} = L_g L_h$ for $(g, h) \in G^2$.

We now discuss properness and G -compactness for locally compact groupoid actions (on G -spaces). Proper actions for locally compact groups have been investigated by R. S. Palais ([24]) and N. C. Phillips ([25]). The basic results for proper locally compact group actions extend to the case of proper locally compact groupoid actions. Specifically, the action of a locally compact groupoid G on a G -space X gives the orbit equivalence relation on X . Further, if X is a proper G -space, then the space X/G is locally compact Hausdorff in the quotient topology, and the quotient map is open ([19, 20]).

The G -space X is called a *proper* G -space if the (continuous) map α , given by $(g, x) \rightarrow (gx, x)$, from $G * X$ into $X \times X$, is a proper map, i.e. if $\alpha^{-1}(C)$ is compact in $G * X$ whenever C is compact in $X \times X$. It is well-known, and easy to check, that

G itself as a G -space is proper. The G -space $X_1 * X_2$ is proper if both G -spaces X_1, X_2 are.

The G -space G^0 is not usually proper (even when G is a Lie group). However, if G^0 is a proper G -space – and this is an assumption of Theorem 1.1 – then *every* G -space is proper.

The G -space X is called *G -compact* (cf. [25, p.23]) if the quotient space X/G is compact. Following the proof of [25, Lemma, p.23-24], we obtain that *if X is a proper G -manifold, then X is G -compact if and only if there exists a compact subset B of X such that $X = GB$.*

One consequence of the existence of a G -compact, proper G -space X is that G^0 as a G -space is also G -compact. For then there is a canonical, surjective, continuous map from X/G onto G^0/G , so that the latter is compact if the former is. In this paper, we will assume that G^0 is G -compact. (This implies that the G -space X of the theorem of this paper is itself G -compact.)

We now discuss invariant averaging and G -partition of unities for a proper G -space. The discussion generalizes the approach of Phillips ([25, Chapter 2]). (For earlier special cases of this, see [7, 11].)

To deal with the varying fibers that arise in groupoid contexts and also with the vector bundle context, we use the following technical lemma. This is a variant of the lemma [22, Lemma C.0.2] (the main idea of which goes back to Renault and Connes). The proof of the lemma is the same *mutatis mutandis* as that of [22, Lemma C.0.2].

LEMMA 3.1. *Let (X, p) be a proper G -space and (\mathcal{E}, ρ) be a vector bundle over X . Let $f' : G *_r X \rightarrow \mathcal{E}$ be continuous and such that $f'((g, x)) \in \mathcal{E}^x$ for each $x \in X$. Suppose further that for each $C \in \mathcal{C}(X)$ and with*

$$W_C = \{(g, x) \in G *_r X : x \in C\},$$

the restriction $f' |_{W_C} \in C_c(W_C, \mathcal{E})$. Then the fiber preserving map $F : X \rightarrow \mathcal{E}$, where

$$(3.1) \quad F(x) = \int f'(g, x) d\lambda^{p(x)}(g)$$

is continuous on X .

The scalar version of the above lemma is the case where \mathcal{E} is the trivial bundle $X \times \mathbb{C}$: in that case, the function f' is scalar valued. If, in addition, $X = G$, then the result is equivalent to [22, Lemma C.0.2].

The scalar version of Lemma 3.1 enables us, in the presence of properness, to average a $C_c(X)$ -function into an invariant continuous function on X . The same argument applies more generally to sections of Banach G -bundles, cf. [25, Lemma 2.4].

A function $f : X \rightarrow \mathbb{C}$ is called *invariant* if for all $x \in X$ and all $h \in G^{p(x)}$, we have

$$f(h^{-1}x) = f(x).$$

PROPOSITION 3.2. *Let X be a proper G -space and $\xi \in C_c(X)$. Then the function $\eta : X \rightarrow \mathbb{C}$, where*

$$(3.2) \quad \eta(x) = \int_{G^{p(x)}} \xi(g^{-1}x) d\lambda^{p(x)}(g).$$

is invariant and continuous.

PROOF. Apply the scalar version of Lemma 3.1 with

$$f'(g, x) = \xi(g^{-1}x)$$

to obtain that η is well-defined and continuous. For invariance, we have, using (2.1):

$$\begin{aligned} \eta(h^{-1}x) &= \int \xi(g^{-1}h^{-1}x) d\lambda^{s(h)}(x) \\ &= \int \xi((hg)^{-1}x) d\lambda^{s(h)}(x) \\ &= \int \xi(k^{-1}x) d\lambda^{p(x)}(k) \\ &= \eta(x). \end{aligned}$$

□

Let X be a proper G -space and $\{U_\alpha\}$ be a collection of open subsets of X . A G -partition of unity of X subordinate to $\{U_\alpha\}$ (cf. [25, p.25]) is a collection of continuous non-negative functions f_γ with compact supports in the U_α 's such that for every $x \in X$, we have

$$(3.3) \quad \sum_{\gamma} \int_{G^{p(x)}} f_\gamma(g^{-1}x) d\lambda^{p(x)}(g) = 1$$

where the sum is locally finite.

PROPOSITION 3.3. *Let X be a proper G -space and $\{U_\alpha\}$ be a collection of open subsets of X such that the sets GU_α cover X . Then there exists a G -partition of unity subordinate to the collection $\{U_\alpha\}$.*

PROOF. We follow the proof of the group version of the result of Phillips in [25, Lemma 2.6]. Let $\pi : X \rightarrow X/G$ be the quotient map. Since X is a proper G -space, the space X/G is a locally compact Hausdorff, second countable space and π is open. The group argument of Phillips then goes through to give continuous functions $u_\gamma : X \rightarrow [0, 1]$ ($\gamma \in S$) with compact supports in the U_α 's and the appropriate local finiteness properties, and for which the function f_1 on X given by

$$f_1(x) = \sum_{\gamma \in S} \int u_\gamma(g^{-1}x) d\lambda^{p(x)}(g)$$

is (by Proposition 3.2) strictly positive and continuous. One then takes $f_\gamma = u_\gamma / f_1$. □

PROPOSITION 3.4. *Let X be a proper, G -compact G -space. Then there exists a non-negative function $c \in C_c(X)$ such that for each $x \in X$, we have*

$$(3.4) \quad \int_{G^{p(x)}} c(g^{-1}x) d\lambda^{p(x)}(g) = 1.$$

PROOF. Let $Q : X \rightarrow X/G$ be the quotient map. Since Q is open and X/G is compact, there exists a compact subset C of X such that $Q(C) = X/G$. Let ξ be in $C_c(X)$ such that $0 \leq \xi \leq 1$ and $\xi(C) = \{1\}$. Define η as in (3.2) and take $c = \xi / \eta$. □

The smooth version of a G -space is a G -manifold (X, p) . For this, G is now a Lie groupoid and X is a smooth manifold. The map $p : X \rightarrow G^0$ is required to be a submersion. Then each X^u is a submanifold of X of fixed dimension j . As in the case of G , there is a basis for the topology of X consisting of *basic* open sets $V \sim p(V) \times Y$ where Y is an open subset of \mathbb{R}^j .

Since the map $s \times p : G \times X \rightarrow G^0 \times G^0$ is also a submersion, the space $G * X = (s \times p)^{-1}(\Delta)$ is a submanifold of $G \times X$, where Δ is the diagonal $\{(u, u) : u \in G^0\}$. Under the above assumptions, the pair (X, p) is called a G -manifold if it is a G -space such that the multiplication map $m : G * X \rightarrow X$ is smooth and inversion is a diffeomorphism.

Note that the pairs $(G, r), (G^0, id)$ are G -manifolds. Also, if (X_i, p_i) ($i = 1, 2$) are G -manifolds, then $(X_1 * X_2, \tilde{p})$ is also a G -manifold.

If (X, p) is a G -manifold, then for each $g \in G$, the map $l_g : X^{s(g)} \rightarrow X^{r(g)}$ is a diffeomorphism, with inverse $l_{g^{-1}}$. The (restricted) map $L_g : C_c^\infty(X^{r(g)}) \rightarrow C_c^\infty(X^{s(g)})$ is an isomorphism of $*$ -algebras.

In this paper, hermitian metrics are assumed conjugate linear in the first variable and linear in the second. It will follow from Proposition 4.2, with $\tilde{E} = TX$, that there is a G -isometric hermitian metric $\langle \cdot, \cdot \rangle$ on the (complex) tangent bundle TX , the complexification of the vector bundle of vectors tangent along the fibers X^u 's. (So in this paper, TX is *not* the (complex) tangent bundle to the manifold X .) This gives in the standard way a family of smooth measures μ^u on the X^u 's. So in terms of local coordinates, we have $d\mu^u = |\det g_{ij}^u|^{1/2} dx$, where the hermitian metric $\langle \cdot, \cdot \rangle$ restricts over X^u to $\langle \cdot, \cdot \rangle^u$, and the latter is given locally by $\sum g_{ij}^u dx_i \otimes dx_j$. The G -isometric property of $\langle \cdot, \cdot \rangle$ just says that the differential $(\ell_g)_* : TX^{s(g)} \rightarrow TX^{r(g)}$ is an isometry. This in turn translates to the G -invariance of the μ^u 's: for $f \in C_c(X^{r(g)})$, we have

$$(3.5) \quad \int_{X^{s(g)}} f(gx) d\mu^{s(g)}(x) = \int_{X^{r(g)}} f(y) d\mu^{r(g)}(y).$$

The counterparts to (i) and (ii) in the definition of a *left Haar system* (§2) also hold.

Note that the C^∞ -versions of Lemma 3.1, Proposition 3.2, Proposition 3.3 and Proposition 3.4 hold for a G -manifold X – the proof modifications are simple.

4. Pseudodifferential operators on G -manifolds

For the rest of the paper, we will assume that G is a Lie groupoid, G^0 is a proper, G -compact, G -manifold, and (X, p) is a G -manifold which is, in addition, a fiber bundle over G^0 with smooth compact fiber Z and structure group $\text{Diff}(Z)$. Consideration of this situation is motivated by the Atiyah-Singer families theorem, which can be interpreted as the case in which X is compact and $G = G^0$, a groupoid of units. (However, in the Atiyah-Singer situation, the base space G^0 is not assumed to be a manifold, and the fiber bundle is not assumed to be a manifold but rather a *manifold over* G^0 . The “continuous family” version of the theorem of this paper requires continuous family groupoids ([23]) and will be considered elsewhere.)

Note that the existence of such a G -manifold imposes conditions on the Lie groupoid G . For example, since the G -action is proper and every X^u is compact, it follows that for every $u, v \in G^0$, the set $\{(g, x) \in G * X : (gx, x) \in X^v \times X^u\}$ is compact. So the sets $G_u \cap G^v$ are compact. In particular, the isotropy groups of

G are compact. It is left to the reader to check that X is G -compact if and only if G^0 is G -compact and X is proper if and only if G^0 is proper. (So in theorem of this paper, X is automatically proper and G -compact.) Note that by replacing μ^u by $\mu^u/\|\mu^u\|$, we can suppose that each μ^u is a probability measure.

We now discuss invariant families of pseudodifferential operators on X . It is natural to impose the condition on the symbols of such operators that they belong to one of the Hörmander class $L_{\rho,\delta}^m$ (as Kasparov does in [11]). However, as Atiyah and Singer comment ([1, p.508]), *for our purposes, any one of these classes would be equally good*. The reason is that for the purposes of the analytic index, the class of operators (associated with the class of symbols taken) gets closed up in a C^* -algebra, and this closure is the same for reasonable choices of starting class. We use the class of pseudodifferential operators in [1], and for the convenience of the reader, will now recall some of the details about these operators ([1],[32, Ch. 4]).

Let B be an open subset of some \mathbb{R}^N and $m \in \mathbb{R}$. Then the space $\tilde{S}^m(B \times \mathbb{R}^N; B \times \mathbb{C})$ of symbols is the set of smooth functions $a : B \times \mathbb{R}^N \rightarrow \mathbb{C}$ such that given multiindices α, β and a compact subset $K \subset B$, there exists a constant C such that

$$(4.1) \quad |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C(1 + |\xi|)^{m-|\alpha|}$$

for all $x \in K, \xi \in \mathbb{R}^N$.

The symbols a that we will be considering are required to satisfy two other properties. Firstly, it is required that for all $x \in B$ and all $\xi \neq 0$ in \mathbb{R}^N , the principle symbol

$$(4.2) \quad \sigma(a)(x, \xi) = \lim_{k \rightarrow \infty} \frac{a(x, k\xi)}{k^m}$$

exists. Clearly, $\sigma(a) : B \times (\mathbb{R}^N \setminus \{0\}) \rightarrow \mathbb{C}$. Secondly, we also require that for any “cut-off” function $\psi \in C^\infty(\mathbb{R}^N)$, i.e. a C^∞ -function on \mathbb{R}^N which is 0 near $\xi = 0$ and 1 outside the unit ball, we have

$$a(x, \xi) - \psi(\xi)\sigma(a) \in \tilde{S}^{m-1}(B \times \mathbb{R}^N; B \times \mathbb{C}).$$

The set of symbols in $\tilde{S}^m(B \times \mathbb{R}^N; B \times \mathbb{C})$ which also satisfy these two properties is denoted by $S^m(B \times \mathbb{R}^N; B \times \mathbb{C})$.

More generally, the space $S^m(B \times \mathbb{R}^N; B \times \mathbb{C}^k)$ is defined in the obvious way by considering functions $a : B \times \mathbb{R}^N \rightarrow \mathbb{C}^k$ whose components are in $S^m(B \times \mathbb{R}^N; B \times \mathbb{C})$. The space $S^m(B \times \mathbb{R}^N; B \times \mathbb{C}^k)$ is a Fréchet space using as seminorms the $\|\cdot\|_{m,B,\alpha,\beta,K}$ ’s, where $\|\cdot\|_{m,B,\alpha,\beta,K}$ is the smallest constant C for which the inequality of (4.1) holds.

Any $a \in S^m(B \times \mathbb{R}^N; B \times M_k(\mathbb{C}))$ determines a pseudodifferential operator $a(x, D_x) : C_c^\infty(B; \mathbb{C}^k) \rightarrow C^\infty(B; \mathbb{C}^k)$, where for $f \in C_c^\infty(B; \mathbb{C}^k)$,

$$(4.3) \quad a(x, D_x)f(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) d\xi.$$

Define the seminorm $\|\cdot\|_{m,B,\alpha,\beta,K}$ on the operators $a(x, D_x)$ by:

$$\|a(x, D_x)\|_{m,B,\alpha,\beta,K} = \|a\|_{m,B,\alpha,\beta,K}.$$

Regarding $B \times \mathbb{R}^N$ as T^*B and replacing $B \times \mathbb{C}^k$ by a smooth complex vector bundle, the above definitions generalize in the standard way to the setting of smooth complex vector bundles E, F over a manifold M . (This is because the class

of pseudodifferential operators above are invariant under diffeomorphisms.) We will take M to be compact but this is not essential. In this setting, an element of $S^m(M; E, F)$ is just a section of the bundle $\text{Hom}(E, F)$ pulled back over the cotangent bundle T^*M . Locally, $\text{Hom}(E, F)$ is of the form $B \times \mathbb{C}^k$ but \mathbb{C}^k is realized as the space of $q \times p$ -matrices, where p, q are the ranks of E and F . So with respect to a coordinate patch on M and bases in E, F , an element of $S^m(M; E, F)$ is just a matrix a_{ij} of symbols of the $S^m(B \times \mathbb{R}^N; B \times \mathbb{C}^k)$ -kind earlier. A pseudodifferential operator

$$D : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

is a map such that for every coordinate neighborhood U of M trivializing E, F , the map that sends $f \in C_c^\infty(U, E)$ to the restriction of $D\psi$ to U is a pseudodifferential operator of the form (4.3), or equivalently ([9, p.84]), that for any $\phi, \psi \in C_c^\infty(U)$, the operator $\phi D\psi$ is a pseudodifferential operator in the sense of (4.3).

The space of such pseudodifferential operators D is denoted by $\Psi^m(M; E, F)$ and is a Fréchet space under the seminorms $D \rightarrow \|(\phi D\psi)\|_{m,U,\alpha,\beta,K}$.

The principle symbol $\sigma(D)$ of D , defined locally in (4.2), is independent of coordinate choices, and is a section of the bundle $S^m(T_0^*M; \text{Hom}(\pi^*E, \pi^*F))$, where T_0^*M is T^*M with the zero section removed and π is the canonical map from T_0^*M to M . An explicit formula for $\sigma(D)$ is:

$$(4.4) \quad \sigma(D)(x, d\phi(x))f(x) = \lim_{k \rightarrow \infty} k^{-m} e^{-ik\phi(x)} D(e^{ik\phi} f)(x)$$

where $\phi \in C^\infty(M)$ and $f \in C^\infty(M, E)$ (e.g. [1, p.509], [7, p.298]).

It is obvious from (4.2) that the principle symbol is homogeneous of degree m , i.e. for $\xi \neq 0$ and $t > 0$, we have

$$a_m(x, t\xi) = t^m a(x, \xi).$$

This homogeneity enables one to regard the principle symbol as a section of the sphere bundle $\text{Hom}(\pi_S^*E, \pi_S^*F)$ over $S(M)$. Here, $S(M)$ is the unit sphere bundle of TM defined by some smooth Riemannian metric on M , $\pi_S : S(M) \rightarrow M$ is the canonical projection, and π_S^*E, π_S^*F are the associated pull-back bundles. Alternatively expressed, $\sigma(D)$ is an element of $\text{Symb}^m(M; E, F)$, the space of smooth homomorphisms from π_S^*E to π_S^*F .

For convenience, from now on, we will only consider the case where $E = F$ and where the order of the pseudodifferential operators involved is 0 . For the first of these, a well-known argument enables us to replace D by $D \oplus D^*$ on $E \oplus F$ (using hermitian metrics on E and F). So we can take $E = F$.

One can reduce to the case $m = 0$ by giving E a hermitian metric and a connection. Then one just replaces D by $(I + (D')^* D')^{-m/2} D$ where D' is the covariant derivative of the connection ([1, p.511]). We will write $\Psi(M; E)$ in place of $\Psi^0(M; E, E)$ and $\text{Symb}(M; E)$ for $\text{Symb}^0(M; E, E)$. An element $D \in \Psi(M; E)$ is called *elliptic* if the section $\sigma(D)$ is invertible.

Let E be given a hermitian metric (so that E becomes a finite-dimensional Hilbert bundle). Every $D \in \Psi(M; E)$ extends to a bounded linear operator on $L^2(M, E, \mu)$ for any smooth positive measure μ on M (e.g. [31, Theorem 6.5]). The closure of $\Psi(M; E)$ in $B(L^2(M, E, \mu))$ will be denoted by $\bar{\Psi}(M; E)$. For $D \in \Psi(M; E)$, let $\tilde{D} \in \bar{\Psi}(M; E)$ be the extension of D by continuity to $L^2(M, E, \mu)$. The following result is due to Atiyah and Singer ([2, p.124], [1, p.512]).

PROPOSITION 4.1. *The map $D \rightarrow \tilde{D}$ is continuous. Further the map $\tilde{D} \rightarrow \sigma(D)$ is continuous, where the norm on $\text{Symb}(M; E)$ is the sup-norm over the unit sphere bundle SM (treating $\pi_S^* E$ as a Hilbert bundle over SM).*

Let (X, p) be (as earlier) a G -manifold which is a smooth fiber bundle over G^0 with fiber Z . Let (\tilde{E}, q) be a smooth complex vector bundle over X in the sense of Atiyah and Singer ([2, p.123]). So \tilde{E} is a vector bundle over X such that $(\tilde{E}, p \circ q)$ is a fiber bundle over G^0 with a fixed smooth complex vector bundle E over Z as fiber and structure group $\text{Diff}(Z, E)$ (the topological group of diffeomorphisms of E that map fibers to fibers linearly). Locally on G^0 , the restrictions $X|_{U \sim U \times Z}$ and $\tilde{E}|_{U \sim U \times E}$.

A natural example of a smooth vector bundle over X is TX .

For $u \in G^0, x \in X$, let $\tilde{E}^{(u)}$ be the restriction of \tilde{E} to X^u and \tilde{E}^x be the x -fiber of \tilde{E} . Note that $\tilde{E}^{(u)}$ is a vector bundle over X^u , while \tilde{E}^x is a vector space. Let $C_c^\infty(X, \tilde{E})$ be the space of smooth sections of \tilde{E} with compact supports. For $f \in C_c(X, \tilde{E})$, let $f^u \in C_c^\infty(X^u, \tilde{E}^{(u)})$ be the restriction of f to X^u .

Equivariance for the smooth fiber bundle \tilde{E} is defined in a way similar to that for group actions (e.g. [29]). Precisely, we require that \tilde{E} be a G -manifold with $q : \tilde{E} \rightarrow X$ a G -map. In addition, each $\ell_g : \tilde{E}^{(s(g))} \rightarrow \tilde{E}^{(r(g))}$, is a vector bundle isomorphism. If \tilde{E} is equivariant, then \tilde{E} is a proper G -space if X is proper.

The fibered product $X * X$ is a fiber bundle over G^0 with fiber $Z \times Z$ and structure group $\text{Diff}(Z \times Z)$. In addition, the external tensor product $\tilde{E} \boxtimes \tilde{E}$ is a smooth vector bundle over $X * X$ with the smooth vector bundle $E \boxtimes E$ as fiber.

PROPOSITION 4.2. *There exists a hermitian metric \langle, \rangle on \tilde{E} which is G -isometric.*

PROOF. (cf. [25, pp.40-41] for the group case.) Let \langle, \rangle' be a hermitian metric on \tilde{E} and c be as in Proposition 3.4. For $\xi, \eta \in \tilde{E}^x$, define

$$\langle \xi, \eta \rangle^x = \int c(g^{-1}x) \langle g^{-1}\xi, g^{-1}\eta \rangle' d\lambda^{p(x)}(g).$$

Applying the scalar version of Lemma 3.1 (to deal with continuity) with $\tilde{E} \boxtimes \tilde{E}$ in place of X and with $f'(g, \xi \otimes \eta) = c(g^{-1}x) \langle g^{-1}\xi, g^{-1}\eta \rangle'$, we obtain a hermitian metric $x \rightarrow \langle, \rangle^x$ for \tilde{E} . Arguing as in the proof of Proposition 3.2 gives that \langle, \rangle is isometric. \square

For the rest of the paper, \tilde{E} (as well as TX) will be assumed to have G -isometric hermitian metrics. The smooth version of the following definition is given in [14, 21].

DEFINITION 4.3. A pseudodifferential family on X is a set $D = \{D^u\}$ ($u \in G^0$) where:

- (1) each $D^u : C^\infty(X^u; \tilde{E}^{(u)}) \rightarrow C^\infty(X^u; \tilde{E}^{(u)})$ belongs to $\Psi(X^u; \tilde{E}^{(u)})$;
- (2) given a basic open subset $V \sim p(V) \times Y$ of X , trivializing \tilde{E} (so that $\tilde{E}|_V \sim p(V) \times Y \times \mathbb{C}^k$), there exists a continuous function $a : p(V) \rightarrow S(T^*Y; Y \times M_k)$ such that for each $u \in p(V)$, and identifying $V \cap X^u$ with Y , we have

$$(4.5) \quad D_u|_{V \cap X^u} = a^u(x, D^x)$$

where $a^u(x, \xi) = a(u)(x, \xi)$.

Let $\Psi(X; \tilde{E})$ be the set of pseudodifferential families above. Definition 4.3 is equivalent to the corresponding definition of a pseudodifferential family given in [2]. There, Atiyah and Singer define H to be the closed subgroup of $\text{Diff}(Z, E) \times \text{Diff}(Z, E)$ consisting of pairs (Ψ, Φ) where Ψ, Φ determine the same element of $\text{Diff}(Z)$. They show that $\Psi(X, \tilde{E})$ is a fiber bundle over G^0 with fiber $\Psi(Z; E)$ and structure group H . Then D is a pseudodifferential family if and only if the map $u \rightarrow D^u$ is a continuous section of $\Psi(X, \tilde{E})$.

DEFINITION 4.4. (cf. [5]) A function $f \in C(X, \tilde{E})$ is said to belong to $C^{0,\infty}(X, \tilde{E})$ if in local terms, the map $u \rightarrow f^u$ from G^0 into $C^\infty(Z, E)$ is continuous (where $C^\infty(Z, E)$ has its standard topology of uniform convergence on compacta for all derivatives). The space of functions f in $C^{0,\infty}(X, \tilde{E})$ that have compact support is denoted by $C_c^{0,\infty}(X, \tilde{E})$.

It is obvious that $C_c^{0,\infty}(X, \tilde{E})$ is a complex vector space which contains $C_c^\infty(X, \tilde{E})$. The space $C_c^{0,\infty}(X, \tilde{E})$ plays a similar role for the family $D = \{D_u\}$ as does $C_c^\infty(M, E)$ does for pseudodifferential operators on manifolds.

PROPOSITION 4.5. *Let D be a pseudodifferential family on X . Then $D : C^{0,\infty}(X, \tilde{E}) \rightarrow C^{0,\infty}(X, \tilde{E})$, where for $x \in X^u$, we define*

$$Df(x) = D^u(f^u)(x).$$

PROOF. Using a partition of unity argument (cf. [8, p.23]), we can suppose that there is a basic open set $V \sim p(V) \times Y$ in X trivializing \tilde{E} and functions $\phi, \psi \in C_c^\infty(V)$ such that $D = \phi D \psi$. Since $u \rightarrow (\psi f)^u$ is continuous from G^0 into $C_c^\infty(Y, \mathbb{C}^k)$ and D is a continuous family, it follows by the joint continuity of pseudodifferential operators on C_c^∞ functions ([9, Theorem 18.1.6]) that $Df \in C_c^{0,\infty}(X, \tilde{E})$. \square

DEFINITION 4.6. A *parametrix* for $D \in \Psi(X; \tilde{E})$ is a pseudodifferential family $P = \{P^u\}$ such that for each u , P^u is a parametrix for D^u , i.e. each of $D^u P^u - I, P^u D^u - I$ is a “smoothing” operator, an element $T \in \Psi(X^u; \tilde{E}^u)$ for which $\sigma(T) = 0$.

PROPOSITION 4.7. *Let D be an elliptic, pseudodifferential family on X . Then there exists a parametrix P for D .*

PROOF. Note that SX is a manifold over G^0 with fiber SZ , and that $\text{End}(\pi_S^*(\tilde{E}))$ is a smooth vector bundle over SX . The vector bundle $\text{End}(\pi_S^*(\tilde{E}))$ restricts on each X^u to $\pi_{S,u}^*(X^u; \text{End}(\tilde{E}^{(u)}))$, where $\pi_{S,u} : SX^u \rightarrow X^u$ is the canonical projection. The “principle symbol” $\sigma(D)$ of D , where $\sigma(D)(u) = \sigma(D^u)$, is a section of the vector bundle $\text{End}(\pi_S^*(\tilde{E}))$. In fact, $\sigma(D)$ is continuous – this follows from the continuity of the maps $u \rightarrow D^u \rightarrow \sigma(D^u)$.

By ellipticity, the symbol $\sigma(D)^{-1}$ exists. In addition, straight-forward computations show that, locally, the matrices $b(u, x, \xi)$ belong to S^0 , and that $u \rightarrow b^u$ is continuous. Following the argument of [32, Ch. IV, Theorem 3.15] (which deals with the case of a single pseudodifferential operator, i.e. where G^0 is a singleton), using basic open sets for an open cover, one constructs a pseudodifferential family P on X whose symbol is $\sigma(D)^{-1}$. Then ([32, Ch. IV, Theorem 4.4]) each P^u is a parametrix for D^u . So P is a parametrix for D . (It can be shown that a product of two pseudodifferential families is a pseudodifferential family (cf. [32, Theorem 3.16]) but we won’t need this.) \square

5. Construction of the analytic index

We now consider invariant pseudodifferential families on the G -manifold X . To this end, there is a section action $g \rightarrow L_g$ of G on $C_c^\infty(X, \tilde{E})$, where $L_g : C^\infty(X^{s(g)}, \tilde{E}^{(s(g))}) \rightarrow C^\infty(X^{r(g)}, \tilde{E}^{(r(g))})$ is the diffeomorphism given by:

$$(5.1) \quad L_g f(x) = g[f(g^{-1}x)] \quad (x \in X^{r(g)}).$$

There is a natural (algebraic) action of the groupoid G on $\Psi(X, \tilde{E})$ given by: $g \rightarrow L_g D^{s(g)} L_{g^{-1}}$. Note that $L_g D^{s(g)} L_{g^{-1}}$ belongs to $\Psi(X^{r(g)}, E^{r(g)})$ by [10].

DEFINITION 5.1. The pseudodifferential family D is called *invariant* if

$$(5.2) \quad L_g D^{s(g)} L_{g^{-1}} = D^{r(g)}.$$

The above notion is well-known in the literature (e.g. [5, 6, 17, 21]). We now discuss the symbol $\sigma(D)$ of an invariant pseudodifferential family D .

Since D is invariant then its symbol $\sigma(D)$ is also invariant under the natural action of G on the sections of $\pi_S^* \text{End}(\tilde{E})$. The action of G on the sections of this bundle is given by: $g \rightarrow g\alpha g^{-1}$. The map $D \rightarrow \sigma(D)$ is equivariant. One easy way to prove this is just to calculate the symbol of $g\sigma(D^{s(g)})g^{-1}$ using the explicit formula (4.4) and the fact that ℓ_g is a diffeomorphism from $X^{s(g)}$ onto $X^{r(g)}$. The details are left to the reader.

Suppose now that D is, in addition, elliptic, and let P be a parametrix for D (Proposition 4.7). There is actually an invariant parametrix P_1 for D . This can be proved by taking

$$P_1^u f(x) = \int_{G^u} c(g^{-1}x)(L_g P^{s(g)} L_{g^{-1}}) f(x) d\lambda^u(g)$$

where $p(x) = u$, $f \in C^\infty(X^u, \tilde{E}^{(u)})$, and c is smooth and as in Proposition 3.4. In this connection, cf. [7, 11, 25]. However, rather than giving the details of the proof of the existence of an invariant parametrix P_1 , it is more convenient to deal with the corresponding question at the Fredholm level later (Proposition 5.4), where the corresponding proof in the group case by Phillips adapts easily.

For the rest of this paper, D is an invariant, elliptic pseudodifferential family on X as above. The analytic index of D will be constructed by adapting the approach of N. C. Phillips in [25] to equivariant K-theory for proper actions. (This in turn was motivated by the work of Segal ([29, 30]).)

We first construct a Hilbert bundle $(\mathfrak{H}, \tilde{p})$ over G^0 . By definition ([25, p.7]), a Hilbert bundle is a locally trivial fiber bundle with a Hilbert space H as fiber and with structure group $U(H)$, the unitary group of H with the strong operator topology. The fiber over $u \in G^0$ of this bundle is $L^2(X^u, \tilde{E}^{(u)}, \mu^u)$. The map \tilde{p} just takes any $f \in L^2(X^u, \tilde{E}^{(u)}, \mu^u)$ to u .

PROPOSITION 5.2. *Let μ be a smooth measure on G^0 and give E a hermitian metric \langle, \rangle . Then, in a natural way, the bundle \mathfrak{H} is a Hilbert bundle with fiber $L^2(Z, E, \mu)$, and it admits a canonical continuous unitary action of the Lie groupoid G .*

PROOF. We will establish the Hilbert bundle structure of \mathfrak{H} by constructing a cocycle $\{g_{\alpha\beta}\}$ for that bundle (e.g. [4, p.48]). Let $\{(U_\alpha, \phi_\alpha)\}$ be an open cover of G^0 which trivializes the fiber bundle $(\tilde{E}, \pi \circ p)$. So each ϕ_α is a fiber preserving

homeomorphism from $\tilde{E}|_{U_\alpha}$ onto $U_\alpha \times E$ which is a vector bundle isomorphism on fibers. For each u , let $\tau_\alpha^u : X^u \rightarrow Z$ be the diffeomorphism determined by ϕ_α . Let $x \in X$, and set $p(x) = u$, $\tau_\alpha^u(x) = z$. Let ϕ_α^u be the restriction of ϕ_α to $\tilde{E}^{(u)}$. Then there exists a positive definite, invertible element $A_\alpha^x \in \text{End}(E^z)$ such that for all $\xi \in \tilde{E}^x$, we have $\langle \xi, \xi \rangle^x = \langle A_\alpha^x \phi_\alpha(\xi), \phi_\alpha(\xi) \rangle^z$, and the map $x \rightarrow A_\alpha^x$ is continuous. Define $\chi_\alpha : \mathfrak{H}|_{U_\alpha} \rightarrow U_\alpha \times L^2(Z, E, \mu)$ by:

$$\chi_\alpha f^u(z) = r_\alpha^u(z)^{1/2} (A_\alpha^x)^{1/2} (f^u \circ (\phi_\alpha^u)^{-1})$$

where $r_\alpha^u = d((\tau_\alpha^u)^* \mu^u)/d\mu$. It is left to the reader to check that the map $\chi_\alpha \chi_\beta^{-1} : U_\alpha \cap U_\beta \rightarrow B(L^2(Z, E, \mu))$ is a cocycle with values in $U(L^2(Z, E, \mu))$. Then \mathfrak{H} is the Hilbert bundle constructed in the standard way with transition functions $g_{\alpha\beta} = \chi_\alpha \chi_\beta^{-1}$. To check that $g_{\alpha\beta}$ is continuous into $U(L^2(Z, E, \mu))$ with the strong operator (=weak operator) topology, one uses elementary measure theory.

Turning to the groupoid action on \mathfrak{H} , for each $g \in G$, by the invariance of the μ^u 's and the isometric action of G on \tilde{E} , the map L_g extends from $C(X^{s(g)}, E^{(s(g))})$ to give a unitary element of $B(L^2(X^{s(g)}, \tilde{E}^{(s(g))}, \mu^{s(g)}), L^2(X^{r(g)}, \tilde{E}^{(r(g))}, \mu^{r(g))})$. Clearly, algebraically, \mathfrak{H} is a G -space with unitary action $g \rightarrow L_g$ where L_g has the same formula as in (5.1). It remains to be shown that the product map $(g, f) \rightarrow L_g f$ from $G * \mathfrak{H}$ into \mathfrak{H} is continuous. Suppose then that $g_n \rightarrow g$ in G , $f_n \rightarrow f$ in \mathfrak{H} with $s(g_n) = \tilde{p}(f_n)$ and $s(g) = \tilde{p}(f)$. Translating this into local terms, we can suppose that U, V are open subsets of G^0 , such that $s(g_n), s(g) \in U, r(g_n), r(g) \in V$ and Z, \tilde{E} are trivial over U and V . In addition, we can suppose that $f_n, f \in L^2(Z, E, \mu)$, and, regarding the L_{g_n}, L_g as unitary on $L^2(Z, E, \mu)$, we have to show that $\|L_{g_n} f_n - L_g f\|_2 \rightarrow 0$. Given $\epsilon > 0$, there exists $F \in C_c(Z, E)$ such that $\|F - f\|_2 < \epsilon$. By the continuity of the action of G on X we have $\|L_{g_n} F - L_g F\|_\infty \rightarrow 0$, and so $\|L_{g_n} F - L_g F\|_2 \rightarrow 0$. An elementary triangular inequality argument then shows that $\|L_{g_n} f_n - L_g f\|_2 < \epsilon$ eventually, so that $\|L_{g_n} f_n - L_g f\|_2 \rightarrow 0$ as required. \square

We now recall some facts about morphisms on Hilbert bundles ([25, Chapter 1]). A *morphism* on \mathfrak{H} is a continuous map $T : \mathfrak{H} \rightarrow \mathfrak{H}$ which restricts to a linear map T^u on each fiber \mathfrak{H}^u and is such that the adjoint map T^* on \mathfrak{H} , where $(T^*)^u = (T^u)^*$, is also continuous. The morphism T is called *equivariant* if for each $g \in G$, we have

$$(5.3) \quad L_g T^{s(g)} L_{g^{-1}} = T_{r(g)}.$$

In particular, if T is equivariant, then

$$(5.4) \quad \|T^{s(g)}\| = \|T^{r(g)}\|$$

for all g .

The morphism T is called *bounded* if

$$(5.5) \quad \|T\| = \sup_{u \in G^0} \|T^u\| < \infty.$$

The set of all bounded morphisms on \mathfrak{H} is a unital C^* -algebra $\mathbf{B}(\mathfrak{H})$ in the obvious way under the norm of (5.5).

It follows by the G -compactness of G^0 , the local boundedness of morphisms ([25, Cor. 1.8]) and (5.4) that every equivariant morphism is automatically bounded.

The set of equivariant morphisms of \mathfrak{H} is a unital C^* -subalgebra of $\mathbf{B}(\mathfrak{H})$, and will be denoted by $\mathbf{B}_G(\mathfrak{H})$.

An element $T \in \mathbf{B}(\mathfrak{H})$ is called *compact* if for every compact subset A of G^0 , the set $\{T^u(\xi^u) : \xi^u \in \mathfrak{H}^u, \|\xi^u\| \leq 1, u \in A\}$ has compact closure in \mathfrak{H} . The set of bounded compact morphisms is a closed ideal of $\mathbf{B}(\mathfrak{H})$ and is denoted by $\mathbf{K}(\mathfrak{H})$ (cf. [25, Lemma 1.12]). Similarly, the set $\mathbf{K}_G(\mathfrak{H})$ of equivariant compact morphisms is a closed ideal of $\mathbf{B}_G(\mathfrak{H})$.

Next we introduce the equivariant compact morphisms which correspond to the “rank 1” operators on a Hilbert module. Let $e_1, e_2 \in C_c(X, \tilde{E})$ and define a section h_{e_1, e_2} of $\tilde{E} \boxtimes \tilde{E}$ by:

$$(5.6) \quad h_{e_1, e_2}(x, y) = \int L_g e_1(x) \boxtimes L_g e_2(y) d\lambda^{p(x)}(g).$$

Note that the section h_{e_1, e_2} in (5.6) has compact support and is continuous by the section version of Proposition 3.2.

Now define the operator $T(e_1, e_2)^u$ on \mathfrak{H}^u by:

$$(5.7) \quad T(e_1, e_2)^u(\xi)(x) = \langle \xi, h_{e_1, e_2}(x, \cdot) \rangle$$

where the inner product on the right-hand side of (5.7) is calculated in $\mathfrak{H}^u = L^2(X^u, \tilde{E}^{(u)}, \mu^u)$. Then $T(e_1, e_2)^u$ is a compact operator since it is a kernel operator whose kernel is continuous with compact support. The equation (5.7) can be usefully written:

$$(5.8) \quad T(e_1, e_2)^u(\xi) = \int L_g e_1 \overline{\langle L_g e_2, \xi \rangle} d\lambda^u(g).$$

PROPOSITION 5.3. *The map $T(e_1, e_2) \in \mathbf{K}_G(\mathfrak{H})$, and the span of morphisms of the form $T(e_1, e_2)$ is dense in $\mathbf{K}_G(\mathfrak{H})$.*

PROOF. We first show that $T(e_1, e_2)$ is well-defined and is continuous on \mathfrak{H} . We just need to show this on some trivialization $U \times L^2(Z, E, \mu)$ of \mathfrak{H} . Applying Lemma 3.1 with $X * X$ in place of X , $\mathcal{E} = \tilde{E} \boxtimes \tilde{E}$ and

$$f'(g, x, y) = L_g e_1(x) \boxtimes L_g e_2(y)$$

gives that h_{e_1, e_2} is continuous on $X * X$. It follows that $T(e_1, e_2)$ is continuous on \mathfrak{H} and that for any compact subset A of U (and hence of G^0) the set

$$\{T(e_1, e_2)^u(\xi^u) : \xi^u \in \mathfrak{H}^u, \|\xi^u\| \leq 1, u \in A\}$$

has compact closure in \mathfrak{H} . Next, $T(e_1, e_2)$ is adjointable with adjoint $T(e_2, e_1)$. So $T(e_1, e_2) \in \mathbf{K}(\mathfrak{H})$. We now show that $T = T(e_1, e_2)$ is invariant. This follows from the argument below using (5.8) and (5.3):

$$\begin{aligned} T^{r(g_0)} L_{g_0} \xi &= \int L_g e_1 \overline{\langle L_g e_2, L_{g_0} \xi \rangle} d\lambda^{r(g_0)}(g) \\ &= \int L_{g_0} [L_{g_0^{-1}g} e_1] \overline{\langle L_{g_0^{-1}g} e_2, \xi \rangle} d\lambda^{r(g_0)}(g) \\ &= \int L_{g_0} [L_h e_1] \overline{\langle L_h e_2, \xi \rangle} d\lambda^{s(g_0)}(h) \\ &= L_{g_0} T^{s(g_0)} \xi. \end{aligned}$$

For the last part of the proposition, we have to show that given $R \in \mathbf{K}_G(\mathfrak{H})$ and $\epsilon > 0$, then there exist ξ_i, η_i ($1 \leq i \leq n$) in $C_c(X, \tilde{E})$ such that

$$(5.9) \quad \|R^u - \sum_{i=1}^n T(\xi_i, \eta_i)^u\| \leq \epsilon$$

for all $u \in G^0$. To this end, we adapt the corresponding argument of Phillips in the locally compact group case ([25, pp.92-93]). Restricting R to trivializing subsets of G^0 , there is then an open cover $\{U_u\}$ of G^0 such that for each u , there exist $\xi_{i,u}, \eta_{i,u} \in C_c(X, \tilde{E})$ such that

$$\|R^v - \sum_i (\xi_{i,u} \otimes \eta_{i,u})^v\| < \epsilon$$

for all $v \in U_u$. Here $(\xi_{i,u} \otimes \eta_{i,u})^v(w) = (\xi_{i,u})^v \overline{(\eta_{i,u})^v, w}$ for $w \in L^2(X^v, \tilde{E}^{(v)}, \mu^v)$. Using Proposition 3.3 with G^0 in place of X , there exists a G -partition of unity $\{f_\gamma\}$ subordinate to the cover $\{U_u\}$ of G^0 . By considering terms of the form $f_\gamma^{1/2} \xi_{i,u} \otimes f_\gamma^{1/2} \eta_{i,u}$ and using the G -compactness of G^0 and the invariance of the morphisms involved, there exist $\xi_j, \eta_j \in C_c(X, \tilde{E})$ (j in some finite index set J) such that for all $u \in G^0$,

$$\|h(u)R^u - \sum_j (\xi_j \otimes \eta_j)^u\| \leq \epsilon h(u)$$

where $h = \sum_\gamma f_\gamma$. Then for any $u \in G^0$, we have

$$\begin{aligned} & \|R^u - \int \sum_j L_g(\xi_j \otimes \eta_j) L_{g^{-1}} d\lambda^u(g)\| \\ &= \| \int [h(g^{-1}u)R^u - \sum_j L_g(\xi_j \otimes \eta_j) L_{g^{-1}}] d\lambda^u(g) \| \\ &= \| \int L_g[h(g^{-1}u)R^{g^{-1}u} - \sum_j (\xi_j \otimes \eta_j)^{g^{-1}u}] L_{g^{-1}} d\lambda^u(g) \| \\ &\leq \int \|h(g^{-1}u)R^{g^{-1}u} - \sum_j (\xi_j \otimes \eta_j)^{g^{-1}u}\| d\lambda^u(g) \\ &\leq \int \epsilon h(g^{-1}u) d\lambda^u(g) = \epsilon. \end{aligned}$$

Since $L_g(\xi_j \otimes \eta_j) L_{g^{-1}} = L_g \xi_j \otimes L_g \eta_j$, we obtain (5.9). \square

An element $T \in \mathbf{B}(\mathfrak{H})$ is called *Fredholm* if there exists $S \in \mathbf{B}(\mathfrak{H})$ such that both $ST - I, TS - I \in \mathbf{K}(\mathfrak{H})$.

- PROPOSITION 5.4. (i) Suppose that $T \in \mathbf{B}_G(\mathfrak{H})$ is Fredholm. Then there exists $S \in \mathbf{B}_G(\mathfrak{H})$ such that both $ST - I, TS - I \in \mathbf{K}_G(\mathfrak{H})$.
(ii) The elliptic pseudodifferential family $D = \{D^u\}$ defines, by extending each D^u to $\tilde{D}^u \in B(\mathfrak{H}^u)$ (Proposition 4.1) an invariant Fredholm morphism on \mathfrak{H} .

PROOF. (i) The locally compact group version of this is given in [25, Lemma 3.7]. For each $u \in G^0$, define $A^u \in B(\mathfrak{H}^u)$ by: for $\xi \in \mathfrak{H}^u$, $x \in X^u$, set

$$(5.10) \quad A^u \xi(x) = \int_{G^u} c(g^{-1}x) L_g S^{s(g)} L_{g^{-1}} f(x) d\lambda^u(g)$$

where c is as in Proposition 3.4. The argument of Phillips adapts directly to show that $A \in \mathbf{B}_G(\mathfrak{H})$ and satisfies $TA - I, AT - I \in \mathbf{K}_G(\mathfrak{H})$.

(ii) By Proposition 4.1, in local terms, the map $u \rightarrow \tilde{D}^u$, regarded as a map into $B(L^2(Z, E, \mu))$, is strong operator – even norm – continuous, and has an adjoint $u \rightarrow (\tilde{D}^u)^*$. So D , identified with $\{\tilde{D}^u\}$, is a morphism. The invariance of this morphism

follows from the invariance of each D^u on the dense subspace $C^\infty(X^u, E^{(u)})$ of $L^2(X^u, \tilde{E}^{(u)}, \mu^u)$. \square

We now construct the Fredholm module which will give the index of the elliptic pseudodifferential family D . The proof constructs a certain Kasparov $(\mathbb{C}, C_{red}^*(G))$ module. The proof in the locally compact group case is effectively given by Phillips ([25, Ch. 6]) in his discussion of the generalized Green-Rosenberg theorem. Actually, Phillips (*loc. cit.*) proves more than that, showing that his equivariant K-theory group $K_G^0(X)$ is isomorphic as an abelian group to $K_0(C^*(G, X))$. We will not consider the groupoid version of this in this paper but consider only the corresponding Kasparov $(\mathbb{C}, C_{red}^*(G))$ -module that gives the analytic index.

Let $\Gamma_c(\mathfrak{H}) = C_c(G^0, \mathfrak{H})$. For $T \in \mathbf{B}_G(\mathfrak{H})$ and $f \in \Gamma_c(\mathfrak{H})$, define a section $\Phi(T)(f)$ of \mathfrak{H} by:

$$(5.11) \quad \Phi(T)(f)(u) = T^u f \quad (= T^u(f(u))).$$

PROPOSITION 5.5. *Let $f \in \Gamma_c(\mathfrak{H})$. Then the section $\Phi(T)(f)$ belongs to $\Gamma_c(\mathfrak{H})$.*

PROOF. Trivially, the support of $\Phi(T)$ is compact in G^0 . Let $u \in G^0$ and $u_n \rightarrow u$ in G^0 . Trivializing in a neighborhood of u in G^0 , we can regard $f(u_n), f(u) \in L^2(Z, E, \mu)$ (cf. Proposition 5.2), and by the continuity of f , $\|f(u_n) - f(u)\|_2 \rightarrow 0$. Since T is continuous on \mathfrak{H} , we have $T_{u_n} f \rightarrow T_u f$. So $\Phi(T)(f)$ belongs to $\Gamma_c(\mathfrak{H})$. \square

We will show that $\Gamma_c(\mathfrak{H})$ is a pre-Hilbert module over the pre- C^* -algebra $C_c(G)$, with $C_c(G)$ -inner product and module operations given below, where $g \in G$, $x \in X$, $e, e_1, e_2 \in \Gamma_c(\mathfrak{H})$ and $f \in C_c(G)$:

$$(5.12) \quad \langle e_1, e_2 \rangle(g) = \int_{X^{r(g)}} \overline{\langle e_1(x), L_g e_2(x) \rangle} d\mu^{r(g)}(x)$$

$$(5.13) \quad ef(x) = \int_{G^{p(x)}} L_g e(x) f(g^{-1}) d\lambda^{p(x)}(g).$$

The preceding equality can alternatively be written as:

$$(5.14) \quad (ef)(u) = \int_{G^u} (L_g e) f(g^{-1}) d\lambda^u(g).$$

The following equality is useful:

$$(5.15) \quad \langle e_1, L_g e_2 \rangle = \langle L_{g^{-1}} e_1, e_2 \rangle$$

where we have omitted the superscripts $r(g), s(g)$ respectively on the two preceding inner products. This follows from (3.5) and the isometric action of G on \tilde{E} .

Note that (5.12) can then be rewritten succinctly as:

$$(5.16) \quad \langle e_1, e_2 \rangle(g) = \overline{\langle e_1, L_g e_2 \rangle}.$$

PROPOSITION 5.6. *The space $\Gamma_c(\mathfrak{H})$ is a pre-Hilbert module over the pre- C^* -algebra $C_c(G) \subset C_{red}^*(G)$ with $C_c(G)$ -inner product and module action given by (5.12) and (5.13) respectively.*

PROOF. We claim first that $\langle e_1, e_2 \rangle \in C_c(G)$. That $\langle e_1, e_2 \rangle$ is continuous follows using (5.16), the continuity of the sections e_1, e_2 and the strong operator continuity of the groupoid action $g \rightarrow L_g$ on \mathfrak{H} . Now let C_i be the (compact) support of the sections e_i . By the properness of the action of G on X , it follows that the set $\{(g, x) : (g^{-1}x, x) \in C_2 \times C_1\}$ is compact, and the support of $\langle e_1, e_2 \rangle$ is contained

in the projection of that set onto the first coordinate. A similar argument (using (5.14)) shows that $ef \in \Gamma_c(\mathfrak{H})$.

It remains to check ([3, p.126]) that (a) \langle, \rangle is sesquilinear, and that for all $e, e_1, e_2 \in \Gamma_c(\mathfrak{H})$ and all $f \in C_c(G)$, we have (b) $\langle e_1, e_2 f \rangle = \langle e_1, e_2 \rangle f$, (c) $\langle e_1, e_2 \rangle^* = \langle e_2, e_1 \rangle$, and that (d) $\langle e, e \rangle \geq 0$, and is 0 if and only if $e = 0$. (a) is obvious. We prove the others in turn.

For (b), using (2.2):

$$\begin{aligned}
 \langle e_1, e_2 f \rangle(g) &= \int \overline{\langle e_1(x), L_g(e_2 f)(x) \rangle} d\mu^{r(g)}(x) \\
 &= \int \overline{\langle e_1(x), g[e_2 f(g^{-1}x)] \rangle} d\mu^{r(g)}(x) \\
 &= \int \int \overline{\langle e_1(x), g[L_k(e_2)(g^{-1}x)] \rangle} f(k^{-1}) d\lambda^{s(g)}(k) d\mu^{r(g)}(x) \\
 &= \int \int \overline{\langle e_1(x), L_{gk}e_2(x) \rangle} f(k^{-1}) d\lambda^{s(g)}(k) d\mu^{r(g)}(x) \\
 &= \int \int \overline{\langle e_1(x), L_h e_2(x) \rangle} f(h^{-1}g) d\lambda^{r(g)}(h) d\mu^{r(g)}(x) \\
 (5.17) \quad &= \int \langle e_1, e_2 \rangle(h) f(h^{-1}g) d\lambda^{r(g)}(h) \\
 &= \langle e_1, e_2 \rangle f(g).
 \end{aligned}$$

For (c), using (5.15) and (5.16), we have $\langle e_2, e_1 \rangle^*(g) = \overline{\langle e_2, L_{g^{-1}}e_1 \rangle} = \overline{\langle L_g e_2, e_1 \rangle} = \langle e_1, e_2 \rangle(g)$.

For (d), from the definition of $C_{red}^*(G)$ in §2, we just have to show that for $u \in G^0$, we have $\pi_u(\langle e, e \rangle) \geq 0$ where π_u was defined in (2.4). Taking $f = \langle e, e \rangle$, we get, by [22, (3.42)], that for $\xi \in C_c(G)$

$$\begin{aligned}
 \langle \pi_u(f)\xi, \xi \rangle &= \int \int f(gh)\xi(h^{-1})\overline{\xi(g)} d\lambda^u(h)\lambda_u(g) \\
 &= \int \int \overline{\langle e, L_{gh}e \rangle} \xi(h^{-1})\overline{\xi(g)} d\lambda^u(h)d\lambda_u(g) \\
 &= \int \int \overline{\langle L_{g^{-1}}e, L_{h^{-1}}e \rangle} \xi(h)\overline{\xi(g)} d\lambda_u(h)d\lambda_u(g) \\
 (5.18) \quad &= \left\langle \int \xi(g)L_{g^{-1}}e d\lambda_u(g), \int \xi(h)L_{h^{-1}}e d\lambda_u(h) \right\rangle_{L^2(X^u, \tilde{E}^{(u)}, \mu^u)} \\
 (5.19) \quad &\geq 0.
 \end{aligned}$$

Since for $u \in G^0$, $\langle e, e \rangle(u) = \int \|e(x)\|^2 d\lambda^u(g)$, it follows that $\langle e, e \rangle = 0$ if and only if $e = 0$. \square

It is elementary that the $C_c(G)$ -valued inner product \langle, \rangle on $\Gamma_c(\mathfrak{H})$ extends to a $C_{red}^*(G)$ -valued inner product, also denoted by \langle, \rangle , on the completion $\Gamma(\mathfrak{H})$ of $\Gamma_c(\mathfrak{H})$ under the norm $e \rightarrow \|\langle e, e \rangle\|^{1/2}$, and is a Hilbert $C_{red}^*(G)$ -module. The C^* -algebras of bounded and compact module maps on $\Gamma(\mathfrak{H})$ will be denoted by $\mathbf{B}(\Gamma(\mathfrak{H}))$ and $\mathbf{K}(\Gamma(\mathfrak{H}))$ respectively. In the following theorem, the map Φ actually determines an isomorphism from $\mathbf{B}_G(\mathfrak{H})$ onto $\mathbf{B}(\Gamma(\mathfrak{H}))$ but we don't include the proof since it is not needed for our purposes.

THEOREM 5.7. *The map Φ of Proposition 5.5 determines a C^* -isomorphism from $\mathbf{B}_G(\mathfrak{H})$ into $\mathbf{B}(\Gamma(\mathfrak{H}))$ that takes $\mathbf{K}_G(\mathfrak{H})$ onto $\mathbf{K}(\Gamma(\mathfrak{H}))$.*

PROOF. Trivially, $\Phi(T) : \Gamma_c(\mathfrak{H}) \rightarrow \Gamma_c(\mathfrak{H})$ is linear. We will show first that is continuous with norm $\leq \|T\|$. In particular, it extends by continuity to a bounded linear map on $\Gamma(\mathfrak{H})$.

Let $e \in \Gamma_c(\mathfrak{H})$ and $T \in \mathbf{B}_G(\mathfrak{H})$. It is sufficient to show that

$$(5.20) \quad \|\langle \Phi(T)e, \Phi(T)e \rangle\| \leq \|T\|^2 \|\langle e, e \rangle\|.$$

This will follow if we can show that for u, ξ as in the proof of (d) of Proposition 5.6,

$$(5.21) \quad \langle \pi_u(\langle \Phi(T)e, \Phi(T)e \rangle) \xi, \xi \rangle \leq \|T\|^2 \langle \pi_u(\langle e, e \rangle) \xi, \xi \rangle.$$

Using (5.18), the invariance of T and the fact that (in the proof) $s(g) = u$:

$$\begin{aligned} \langle \pi_u(\langle \Phi(T)e, \Phi(T)e \rangle) \xi, \xi \rangle &= \left\| \int \xi(g) L_{g^{-1}} T^{r(g)} e \, d\lambda_u(g) \right\|_2^2 \\ &= \left\| \int \xi(g) T^u L_{g^{-1}} e \, d\lambda_u(g) \right\|_2^2 \\ &\leq \|T^u\|^2 \left\| \int \xi(g) L_{g^{-1}} e \, d\lambda_u(g) \right\|^2 \end{aligned}$$

and (5.21) follows.

So $\Phi(T)$ is a bounded linear operator on $\Gamma(\mathfrak{H})$ and $\|\Phi(T)\| \leq \|T\|$. To obtain $\Phi(T) \in \mathbf{B}(\Gamma(\mathfrak{H}))$, we show that $\Phi(T)$ is an adjointable module map on $\Gamma(\mathfrak{H})$ with adjoint $\Phi(T^*)$.

For $e_1, e_2 \in \Gamma(\mathfrak{H})$ and $g \in G$, we have using (5.16) and the invariance of T^* ,

$$\begin{aligned} \langle \Phi(T)e_1, e_2 \rangle(g) &= \overline{\langle \Phi(T)e_1, L_g e_2 \rangle} \\ &= \overline{\langle T^{r(g)} e_1, L_g e_2 \rangle} \\ &= \overline{\langle e_1, (T^{r(g)})^* L_g e_2 \rangle} \\ &= \overline{\langle e_1, L_g (T^{s(g)})^* e_2 \rangle} \\ &= \langle e_1, \Phi(T^*) e_2 \rangle. \end{aligned}$$

So $\Phi(T^*)$ is an adjoint for $\Phi(T)$.

Next we have to show that $\Phi(T)$ is a module map. To this end, for $e \in \Gamma_c(\mathfrak{H})$, $f \in C_c(G)$, we have by (5.14),

$$\begin{aligned} [\Phi(T)(ef)](u) &= T^u(ef)(u) \\ &= T^u \int L_g(e) f(g^{-1}) \, d\lambda^u(g) \\ &= \int T^{r(g)} L_g(e) f(g^{-1}) \, d\lambda^u(g) \\ &= \int L_g T^{s(g)}(e) f(g^{-1}) \, d\lambda^u(g) \\ &= (Te)f(u). \end{aligned}$$

So $\Phi(T)$ is a module map. From (5.11), Φ is a $*$ -homomorphism that is one-to-one. So Φ is a C^* -isomorphism from $\mathbf{B}_G(\mathfrak{H})$ into $\mathbf{B}(\Gamma(\mathfrak{H}))$.

For the last part of the theorem, from Proposition 5.3 and the density of $\Gamma_c(\mathfrak{H})$ in $\Gamma(\mathfrak{H})$, we just have to show that $\Phi(T(e_1, e_2)) \in \mathbf{K}(\Gamma(\mathfrak{H}))$. In fact, $\Phi(T(e_1, e_2))$

is just the “rank 1” operator θ_{e_1, e_2} ([3, p.128]). Indeed, for $e \in \Gamma_c(\mathfrak{H})$ and $u \in G^0$, using 5.8:

$$\begin{aligned}
\Phi(T(e_1, e_2))e(u) &= T(e_1, e_2)^u e \\
&= \int L_g e_1 \overline{\langle L_g e_2, e \rangle} d\lambda^u(g) \\
&= \int L_g e_1 \overline{\langle e_2, L_{g^{-1}} e \rangle} d\lambda^u(g) \\
&= \int L_g e_1 \langle e_2, e \rangle (g^{-1}) d\lambda^u(g) \\
&= e_1 \langle e_2, e \rangle (u) \\
&= \theta_{e_1, e_2}(e)(u).
\end{aligned}$$

□

We can now complete the construction of the analytic index of the pseudodifferential operator D . By Proposition 5.4, D is an invariant Fredholm morphism on \mathfrak{H} and there exists an $S \in \mathbf{B}_G(\mathfrak{H})$ such that both $SD - I, DS - I \in \mathbf{K}_G(\mathfrak{H})$. Let $a = \Phi(D), b = \Phi(S)$. By Theorem 5.7, we have $ab - I, ba - I \in \mathbf{K}(\Gamma(\mathfrak{H}))$. We summarize the well-known construction of a Fredholm module from a . Let $\pi : \mathbf{B}(\Gamma(\mathfrak{H})) \rightarrow \mathbf{B}(\Gamma(\mathfrak{H}))/\mathbf{K}(\Gamma(\mathfrak{H}))$ be the quotient map. Then $\pi(a)$ is invertible. Let u be the unitary part of $\pi(a)$ given by the polar decomposition of $\pi(a)$: so $u = \pi(a)[\pi(a^*a)]^{-1/2}$. Let $c \in \mathbf{B}(\Gamma(\mathfrak{H}))$ be such that $\pi(c) = u$. Then $cc^* - I, c^*c - I \in \mathbf{K}(\Gamma(\mathfrak{H}))$. We then get a Kasparov $(\mathbb{C}, C_{red}^*(G))$ -bimodule

$$(\Gamma(\mathfrak{H}) \oplus \Gamma(\mathfrak{H}), \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix})$$

which gives an element of $KK^0(\mathbb{C}, C_{red}^*(G)) = K_0(C_{red}^*(G))$. This element is independent of the choice of c by the invariance of Kasparov classes under compact perturbations. This element is the analytic index of D .

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